Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-15

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. Recall that the gradient vector field $\nabla(y^2 - x^2)$ is perpendicular to level sets of the function $y^2 - x^2$, of which the hyperbola in question is one instance. To verify that $\langle y^2, xy \rangle$ is tangent to the hyperbola, then, it will suffice to take the dot product with $\nabla(y^2 - x^2)$ and verify that it is equal to zero along $y^2 - x^2 = 1$.

Actually it turns out that it is equal to zero everywhere:

$$\langle -2x, 2y \rangle \cdot \langle y^2, xy \rangle = -2xy^2 + 2xy^2 = 0.$$

Question 2. This is a plane: $x + y - z = 0$.

Question 3. This is a parabolic cylinder: $x = y^2$.

Answers to computations

Problem 1. There are a couple of ways to parametrize this triangle. One way is to find a Cartesian equation for the plane containing the triangle, solve for one variable as the dependent variable, and then use the two other independent variables as the parameters. If we proceed as such, we find the plane equation

$$4x - 2y + z = 4$$

which we can rewrite as $z = 4 - 4x + 2y$. As you are familiar with, we then have

$$\mathrm{d}S = \sqrt{1 + 4^2 + 2^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{21}\,\mathrm{d}x\,\mathrm{d}y$$

and the integral becomes

$$\int_0^1 \int_{2x-2}^0 x\sqrt{21}\,\mathrm{d}y\,\mathrm{d}x$$

since the relevant region in the xy-plane is a triangle with vertices $(0,0)$, $(1,0)$, $(0,-2)$.

An alternative approach is to map the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ in the uv-plane onto the triangle of interest via

$$\mathbf{r}(u,v) = \langle 1,0,0 \rangle + u\langle -1,-2,0 \rangle + v\langle -1,0,4 \rangle = \langle 1-u-v, -2u, 4v \rangle.$$

In this situation,

$$\mathrm{d}S = |\langle -1,-2,0 \rangle \times \langle -1,0,4 \rangle|\,\mathrm{d}u\,\mathrm{d}v = |\langle -8,4,-2 \rangle|\,\mathrm{d}u\,\mathrm{d}v = 2\sqrt{21}\,\mathrm{d}u\,\mathrm{d}v$$

and we end up with the integral

$$\int_0^1 \int_0^{1-u} (1-u-v)2\sqrt{21}\,\mathrm{d}v\,\mathrm{d}u.$$

Problem 2. We can parametrize this as $\mathbf{r}(x,y) = \langle x, y, 4 - x^2 - y^2 \rangle$, and

$$\mathbf{r}_x \times \mathbf{r}_y = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} = \langle 2x, 2y, 1 \rangle$$

Note that the z-component is positive, so this does give the upwards orientation as we want. So the integral to compute is just

$$\int_0^1 \int_0^1 \langle xy, y(4-x^2-y^2), x(4-x^2-y^2) \rangle \cdot \langle 2x, 2y, 1 \rangle\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 (-x^3 - 2x^2y^2 + 2x^2y - xy^2 + 4x - 2y^4 + 8y^2)\,\mathrm{d}x\,\mathrm{d}y$$

which is hideous but straightforward to compute.